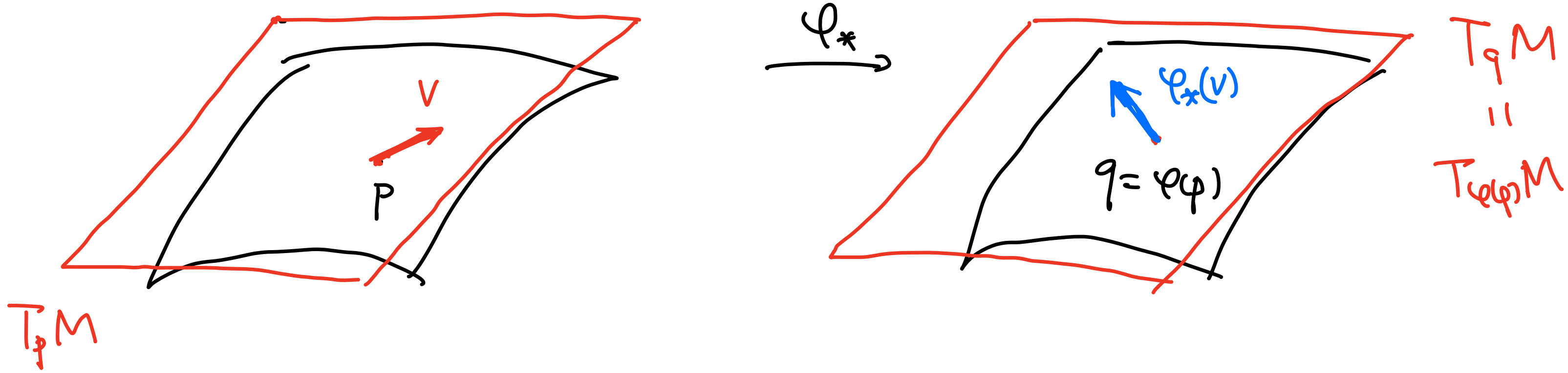
④ Pushforward & pullback

A (single) diffeo φ on M can help us to transfer the tensor field information.

- Pushforward by φ



$$v \longmapsto \varphi_*(v) \quad (\text{or } D\varphi(v))$$

In local coordinate,

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \longmapsto \varphi_*(v) = \underbrace{Jac(\varphi)(p)}_{\text{matrix}} \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

e.g. Suppose $(x_i)_{i=1,\cdots,n}$ is the local coordinate near p

$(x_i')_{i=1,\cdots,n}$ is the local coordinate near $\varphi(p)$

Then for coordinate function x_i',

$$\varphi_*(v)\cdot x_i' = \left(\mathrm{Jac}(\varphi)(p)\cdot\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}\right)\cdot x_i'$$

$$= \sum_{j=1}^{n} \frac{\partial \varphi_i}{\partial x_j} v_j = v\cdot(\underbrace{x_i'\circ\varphi}_{\varphi_i = \varphi_i(x_1,\cdots,x_n)})$$

In general, $\varphi_*(v)\cdot f = v\cdot(f\circ\varphi)$ for any function f on M.

e.g. For two diffeos φ and ψ, $\psi_*\circ\varphi_* = (\psi\circ\varphi)_*$.

$\Rightarrow$ For a diffeo φ, $\varphi_*(p): T_pM \to T_{\varphi(p)}M$ is an isomorphism.

$\Rightarrow$ If $p\in \mathrm{Fix}(\varphi)$, then $\varphi_*(p): T_pM \circlearrowleft$ is an automorphism.

Once a local coordinate near p is fixed, then eigenvalues of $\varphi_*(p)$ characterizes dyn properties of φ near p.

① $(x,y) \overset{\varphi}{\longmapsto} (\cos\theta\, x - \sin\theta\, y,\ \cos\theta\, y + \sin\theta\, x)$ where θ is fixed.

$\varphi(0) = 0$ and $\varphi_*(0) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

② $(x,y) \overset{\varphi}{\longmapsto} (2x+y,\ x+y)$

$\varphi(0) = 0$ and $\varphi_*(0) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ Arnold's cat map

For ①, eigenvalues of $\varphi_*(0)$ lie on the unit circle

For ②, eigenvalues of $\varphi_*(0)$ are <1 and >1

$\Longrightarrow$ For $\varphi_*^n(0)$ as $n \to \infty$, ② will generate a certain chaos.

- Pullback by φ

For $\alpha \in \Gamma(\Lambda^k T^*M) = \Omega^k(M)$, a differential k-form, define

$$(\varphi^*\alpha)_p (v_1, \cdots, v_k) := \alpha_{\varphi(p)} (\varphi_*(v_1), \cdots, \varphi_*(v_k))$$

- φ^* does not change the degree.
- $(\varphi^* f)(p) = f(\varphi(p))$
- $$(\varphi^* (\underset{k}{\alpha} \wedge \underset{l}{\beta}))(v_1, \cdots, v_{k+l}) = (\alpha\wedge\beta)(\varphi_*(v_1), \cdots, \varphi_*(v_{k+l}))$$
$$= (\varphi^*\alpha \wedge \varphi^*\beta)(v_1, \cdots, v_{k+l}).$$

<u>Rmk</u>. For $f \in C^\infty(M)$, $\alpha \in \Omega^k(M)$, we have

$$\varphi^*(\underset{f\wedge\alpha}{\underbrace{f\alpha}}) = (\varphi^* f)(\varphi^*\alpha) = \underline{\underline{(f\circ\varphi)}}(\varphi^*\alpha)$$

A common mistake is

$\otimes$ $\varphi^*(f\alpha) = f(\varphi^*\alpha)$

Rmk For local coordinate $(x_1, \cdots, x_n)$,

$$\varphi^*(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \varphi^* dx_{i_1} \wedge \cdots \wedge \varphi^* dx_{i_k}$$

$$= d\varphi_{i_1} \wedge \cdots \wedge d\varphi_{i_k}$$

$$(\varphi^* dx_i)(v) = dx_i(\varphi_*(v))$$
$$= d(x_i \circ \varphi)(v)$$
$$= d\varphi_i(v).$$

where $\varphi = (\varphi_1, \cdots, \varphi_n)$

$$\Rightarrow \varphi^*(\overbrace{\Sigma f_{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}}^{\text{local expression of } \alpha \in \Omega^k(M)})$$

$$= \Sigma (\varphi^* f_{i_1 \cdots i_k}) d\varphi_{i_1} \wedge \cdots \wedge d\varphi_{i_k}$$

- For any $\alpha \in \Omega^k(M)$, $\varphi^* d\alpha = d(\varphi^* \alpha)$

locally write $\alpha = \Sigma f_{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. Then

$$d(\varphi^* \alpha) = \Sigma\, d(\varphi^* f_{i_1 \cdots i_k}) \wedge d\varphi_{i_1} \wedge \cdots \wedge d\varphi_{i_k} + 0$$ b/c $d \circ d = 0$

$$= \Sigma\, \varphi^*(d f_{i_1 \cdots i_k}) \wedge \varphi^*(dx_{i_1} \wedge \cdots \wedge dx_{i_k})$$

$$= \varphi^*\, d(\Sigma f_{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k})$$

$$= \varphi^* d\alpha.$$

- For a diffeo φ on M. φ^* is an isomorphism (b/c $\psi^* \circ \varphi^* = (\varphi \circ \psi)^*$)

e.g. $\varphi \in \mathrm{Diff}(\mathbb{R}^n)$.

$$\varphi^*(\underbrace{dx_1 \wedge \cdots \wedge dx_n}_{\text{(top) volumn form}}) = d\varphi_1 \wedge \cdots \wedge d\varphi_n$$

$$= \det(\mathrm{Jac}(\varphi))\, dx_1 \wedge \cdots \wedge dx_n$$

If $\det(\mathrm{Jac}(\varphi)) = 1$, then φ is called a volumn-preserving diffeo.

(This also means most diffeos won't preserve volumn).

e.g. φ^* can also act on a general tensor.

- For a Riem mfd (M, g), an isometry is a diffeo on M s.t $\varphi^* g = g$. In particular, an isometry always preserves lengths (given by g) and angles.

⑤ Lie derivative (introduced by Ślebodziński 1931)

- Lie derivative measures how a tensor field T changes along a vector field X

- Lie derivative preserves the tensor type.

- When T = vector field Y

$$\underbrace{L_X}_{\text{vector field}} Y := \lim_{t \to 0} \frac{(\varphi^X_{-t})_* Y - Y}{t}$$

reference v.f.

where φ^X_t is the 1-par group of diffeos associated to the reference v.f. X.

pointwise

$$(\varphi^X_{-t})_*(\varphi^X_t(p)) \, Y_{\varphi^X_t(p)} - Y_p$$

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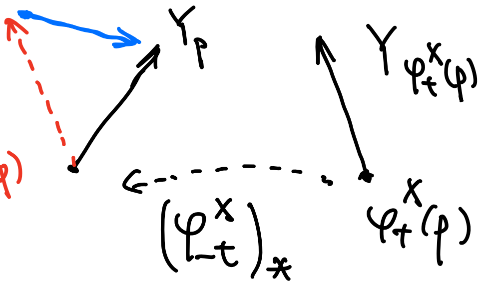
$(\varphi^X_{-t})_*(\varphi^X_t(p))$ acts on $Y_{\varphi^X_t(p)}$



The reason to apply $(\varphi^X_{-t})_*$ is that $Y_{\varphi^X_t(p)}$ and Y_p do not lie in the same tangent space!

Exe: $L_X Y = [X, Y]$

$\Rightarrow$ For $X, Y, Z \in \Gamma(TM)$, we have

$$L_X([Y,Z]) = [L_X Y, Z] + [Y, L_X Z]$$

$\left(\Leftrightarrow [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0\right)$

- When $T = \alpha \in \Omega^k(M)$.

$$\underbrace{L_X \alpha}_{\in \Omega^k(M)} = \lim_{t \to 0} \frac{(\varphi_t^X)^* \alpha - \alpha}{t}$$

where φ_t^X is the 1-par group of diffeo associated to the ref. v.f. X.

Prop (Cartan's magic formula): $L_X \alpha = d(\iota_X \alpha) + \iota_X (d\alpha)$

This is in fact a commutator but with the sign switched.

$\Rightarrow$ ① If $f \in C^\infty(M)$, then

$$L_X f = d\underbrace{\iota_X f}_{=0} + \iota_X df = X\cdot f \ (= df(X))$$

② For any $\alpha \in \Omega^k(M)$,

$$d L_X \alpha = d(\iota_X d\alpha + d\iota_X \alpha) = d\iota_X d\alpha$$

$$= d\iota_X d\alpha + \iota_X(\underbrace{d(d\alpha)}_{=0}) = L_X d\alpha.$$

③ For $\alpha, \beta \in \Omega^*(M)$,

This can also be proved directly from definition.

$$L_X(\alpha\wedge\beta) = \iota_X d(\alpha\wedge\beta) + d\,\iota_X(\alpha\wedge\beta)$$

$$= \iota_X(d\alpha\wedge\beta) + (-1)^{\deg\alpha}\iota_X(\alpha\wedge d\beta) + d(\iota_X\alpha\wedge\beta + (-1)^{\deg\alpha}\alpha\wedge\iota_X\beta)$$

$$= \iota_X d\alpha\wedge\beta + (-1)^{\deg\alpha+1} d\alpha\wedge\iota_X\beta + (-1)^{\deg\alpha}\iota_X\alpha\wedge d\beta + \alpha\wedge\iota_X d\beta$$

$$+ d\iota_X\alpha\wedge\beta + (-1)^{\deg\alpha-1}\iota_X\alpha\wedge d\beta + (-1)^{\deg\alpha} d\alpha\wedge\iota_X\beta + \alpha\wedge d\iota_X\beta$$

$$= L_X\alpha\wedge\beta + \alpha\wedge L_X\beta.$$

④ $L_{fX}\alpha = d\iota_{fX}\alpha + \iota_{fX}d\alpha$

$$= d(f\cdot \iota_X\alpha) + f\,\iota_X d\alpha$$

$$= df\,\iota_X\alpha + f\,d\iota_X\alpha + f\,\iota_X d\alpha = df\cdot\iota_X\alpha + f\cdot L_X\alpha.$$

The proof of Prop above is another typical example of "local argument".

- $$L_X f = \lim_{t\to 0}\frac{(\varphi_t^X)^* f - f}{t} = \lim_{t\to 0}\frac{f\cdot\varphi_t^X - f}{t} \overset{\text{by def}}{=} X\cdot f$$

- $$(L_X df)(v) = \lim_{t\to 0}\frac{(\varphi_t^X)^* df(v) - df(v)}{t} = \lim_{t\to 0}\frac{df((\varphi_t^X)_*(v)) - df(v)}{t}$$

$$= \lim_{t\to 0}\frac{d(f\circ\varphi_t^X)(v) - df(v)}{t}$$

$$= v(Xf) = d(L_X f)(v)$$

- Assume for $\omega\in\Omega^{\leq k}(M)$, Cartan's magic formula holds. Let $\alpha\in\Omega^{k+1}(M)$.

(locally) $\alpha = \sum f_{i_1\cdots i_{k+1}}\, dx_{i_1}\wedge\cdots\wedge dx_{i_{k+1}}$

Rewrite term $f_{i_1 \cdots i_{k+1}} dx_{i_1} \wedge \cdots \wedge dx_{i_{k+1}}$ by

$$dx_{i_1} \wedge \Big(\underbrace{f_{i_1 \cdots i_{k+1}} dx_{i_2} \wedge \cdots \wedge dx_{i_{k+1}}}_{=: \beta \in \Omega^k(M)} \Big)$$

$$\begin{aligned} \mathcal{L}_X (dx_{i_1} \wedge \beta) &= (\mathcal{L}_X dx_{i_1}) \wedge \beta + dx_{i_1} \wedge \mathcal{L}_X \beta \\ &= \underline{d(X \cdot x_{i_1})} \wedge \beta + dx_{i_1} \wedge d\iota_X \beta + dx_{i_1} \wedge \iota_X d\beta \end{aligned}$$

by the previous step

$$\begin{aligned} (d\iota_X + \iota_X d)(dx_{i_1} \wedge \beta) &= d\iota_X (dx_{i_1} \wedge \beta) + \iota_X d(dx_{i_1} \wedge \beta) \\ &= d\big(dx_{i_1}(X) \wedge \beta - dx_{i_1} \wedge \iota_X \beta\big) - \iota_X (dx_{i_1} \wedge d\beta) \\ &= d(X \cdot x_{i_1}) \wedge \beta + dx_{i_1}(X) \wedge d\beta + dx_{i_1} \wedge d\iota_X \beta \\ &\quad - dx_{i_1}(X) \wedge d\beta + dx_{i_1} \wedge \iota_X d\beta \\ &= d(X \cdot x_{i_1}) \wedge \beta + dx_{i_1} \wedge d\iota_X \beta + dx_{i_1} \wedge \iota_X d\beta \end{aligned}$$

$\Rightarrow \mathcal{L}_X = d\iota_X + \iota_X d$ on any $\alpha \in \Omega^{k+1}(M)$. Inductively, done. $\square$